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Functional integrals for parabolic differential equations: II. Time-dependent generators

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Abstract. The convergence of a discretisation procedure for path integrals associated with a class of parabolic second-order differential equation with time-dependent coefficients is shown. The proof is based on a generalisation of the product formula.

1. Introduction

We generalise the results of our previous paper (Alicki and Makowiec 1985, hereafter referred to as AM) for the case of time-dependent coefficients in a differential equation. We consider the following differential equation:

$$\partial f(x;t)/\partial t = \mathscr{L}(x, D; t) f(x;t) \qquad t \in [a, b] \tag{1.1}$$

where $x = (x^1 \ldots x^\nu) \in \mathbb{R}^\nu$, $D = (\partial_1 \ldots \partial_\nu)$, $\partial_\alpha = \partial / \partial x^\alpha$ and

$$\mathscr{L}(x, D; t) = a^{\alpha\beta}(x;t)\partial_\alpha\partial_\beta + b^\alpha(x;t)\partial_\alpha + c(x;t). \tag{1.2}$$

Here the Einstein summation convention is used for Greek letters only. We assume that for all $x \in \mathbb{R}^\nu$, $t \in [a, b]$, $a^{\alpha\beta}(x;t)$ is a strictly positive matrix. We denote by $a_{\alpha\beta}(x;t)$ the inverse of $a^{\alpha\beta}(x;t)$, by $a^{\alpha\beta}_{1/2}(x;t)$ and $a^{1/2}_{\alpha\beta}(x;t)$ the square-root matrices and by $|a(x;t)|$ the determinant of $a^{\alpha\beta}(x;t)$.

The formal path integral representation for the solution of (1.1) may be written as follows.

(i) Phase space form

$$f(x_0; t_0) = \lim_{\substack{N\to\infty \\ \Delta t \to 0}} \int_{\mathbb{R}^{2N\nu}} \prod_{k=1}^{N} \frac{\mathrm{d}x_k\, \mathrm{d}p^k}{(2\pi)^\nu} \exp\bigg[-\sum_{k=1}^{N} \Delta t_k \bigg(-\mathrm{i}p^k_\alpha \frac{\Delta x^\alpha}{\Delta t_k} + a^{\alpha\beta}(x_{k-1}; t_k) p^k_\alpha p^k_\beta + \mathrm{i} b^\alpha(x_{k-1}; t_k) p^k_\alpha - c(x_{k-1}; t_k)\bigg)\bigg] f(x_N; t_N). \tag{1.3}$$

(ii) Configuration space form (obtained by integration over p^k)

$$f(x_0; t_0) = \lim_{\substack{N\to\infty \\ \Delta t \to 0}} \int_{\mathbb{R}^{N\nu}} \prod_{k=1}^{N} \mathrm{d}x_k \prod_{k=1}^{N} (4\pi\Delta t_k)^{-\nu/2} |a(x_{k-1}; t_k)|^{-1/2} \times \exp\bigg\{-\sum_{k=1}^{N} \Delta t_k \bigg[\tfrac{1}{4} a_{\alpha\beta}(x_{k-1}; t_k)\bigg(\frac{\Delta x^\alpha_k}{\Delta t_k} + b^\alpha(x_{k-1}; t_k)\bigg) \times \bigg(\frac{\Delta x^\beta_k}{\Delta t_k} + b^\beta(x_{k-1}; t_k)\bigg) - c(x_{k-1}; t_k)\bigg]\bigg\} f(x_N; t_N), \tag{1.4}$$

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where $a \leqslant s = t_N < t_{N-1} < \ldots < t_1 < t_0 = t \leqslant b$

$$\Delta t_k = t_{k-1} - t_k \qquad \Delta t = \max\{\Delta t_k\} \qquad \Delta x_n^\alpha = x_{k-1}^\alpha - x_k^\alpha .$$

In AM we proved the existence of the limits (1.3) and (1.4) for the case of time independent $a_{\alpha\beta}(x)$, $b^\alpha(x)$, $c(x)$ in the strong topology in different spaces $L^p(\mathbb{R}^\nu)$ and using the equipartition $\Delta t_k = (t-s)/N$. To extend these results we shall prove in the next section a product theorem and in the last section we shall apply it to the class of differential equations of type (1.1) and (1.2) on the space $L^2(\mathbb{R}^\nu)$. The applications of the formal expressions (1.3) and (1.4) to discretisation procedures evaluating functional integrals associated with classical and quantum physics problems are presented, for example, in a book by Langouche *et al* (1982).

2. Generalised product theorem

Let us consider the following evolution equation on a Banach space X:

$$\mathrm{d}f_t/\mathrm{d}t = [Z + L(t)]f_t \tag{2.1}$$

with initial condition f_s, where $(t, s) \in \mathcal{J} = \{(t, s), t \geqslant s, t, s \in [a, b]\}$. We make the following assumptions.

(A) Z is a generator of a strongly continuous one-parameter contraction semigroup on X with domain $\mathcal{D}$ and core $\mathcal{D}_0$.

(B) For $t \in [a, b]$, $L(t)$ defined on $\mathcal{D}_0$ is a dissipative operator and the function $t \to L(t)f \in X$ is strongly continuous for all $f \in \mathcal{D}_0$. Moreover, there exists a strong derivative

$$L'(t)f = (\mathrm{d}/\mathrm{d}t)L(t)f \quad \text{for all } f \in \mathcal{D}_0$$

which is again a strongly continuous function of $t \in [a, b]$.

(C) There exist constants $C_1 < 1$, $C_2 C_3$ such that

$$\|L(t)f\| \leqslant C_1\|Zf\| + C_2\|f\| \qquad \|L'(t)f\| \leqslant C_3 |||f||| \tag{2.2}$$

for all $f \in \mathcal{D}_0$, $t \in [a, b]$ where $|||f||| = \|Zf\| + \|f\|$ is the so-called graph norm.

Remarks.

(1) $\mathcal{D}$ with the graph-norm $|||\ |||$ is a Banach space (Davies 1980).

(2) There exists a unique extension of $Z + L(t)$ to $\mathcal{D}$ (Davies 1980).

(3) There exists a two-parameter family $\{U(t, s), (t, s) \in \mathcal{J}\}$ of linear contractions on X such that

(a) for any partition $s = t_N < t_{N-1} \cdots < t_1 < t_0 = t$

$$U(t, s)f = \lim_{\substack{N \to \infty \\ \Delta t \to 0}} \exp[\Delta t_1(Z + L(t_1))] \ldots \exp[\Delta t_N(Z + L(t_N))]f$$

for all $f \in X$, where $\Delta t = \max_k\{\Delta t_k = t_k - t_{k-1}\}$,

(b) $U(t, s)$ is uniformly strongly continuous in $\mathcal{J}$ and $W(t, s)$ defined by the formula

$$[Z + L(t)]U(t, s)f = W(t, s)[Z + L(s)]f \qquad f \in \mathcal{D}$$

is a bounded operator-valued function uniformly strongly continuous in $\mathcal{J}$,

(c) for $f_s \in \mathcal{D}$ we have $f_t = U(t, s)f_s \in \mathcal{D}$ and

$$\mathrm{d}f_t/\mathrm{d}t = [Z + L(t)]f_t$$

(d) the semigroup property holds:

$$U(t,r)U(r,s)=U(t,s) \qquad \text{for } a\leqslant s\leqslant r\leqslant t\leqslant b$$

(e) for all $f\in\mathscr{D}$ the map $t\to U(t,s)f$ is a strongly continuous function from $[s,b]$ into the Banach space $\mathscr{D}$ (with the norm $|||\ |||$) (Reed and Simon 1975, theorem X.70; Dollard and Friedman 1978, corollary 1 after theorem 12).

We suppose that there exists a two-parameter family of bounded linear operators on X:

$$\{F(t,\tau);\, t\in[a,b],\, \tau\in(0,\delta)\} \qquad F(t,\tau): X\to X$$

such that

(i) $\|F(t,\tau)\|\leqslant e^{\alpha\tau}$ where $\alpha\in\mathbb{R}$, $t\in[a,b]$, $\tau\in(0,\delta)$,

(ii) for any $f\in\mathscr{D}_0$ the following limit exists uniformly in $t\in[a,b]$:

$$\lim_{\tau\to0}\tau^{-1}\{F(t,\tau)f-f\}=\{Z+L(t)\}f$$

(iii) there exists a constant $D<\infty$ such that for all $f\in\mathscr{D}_0$

$$\sup_{\substack{t\in[a,b]\\ \tau\in(0,\delta)}}\tau^{-1}\|F(t,\tau)f-f\|\leqslant D|||f|||.$$

The operator $F(t,\tau)$ approximates the propagator $U(t+\tau,t)$ in the following sense.

Lemma 2.1. Let

$$R(t,\tau)f=\tau^{-1}\{F(t,\tau)f-U(t+\tau,t)f\} \qquad \text{for } f\in\mathscr{D}.$$

Then there exists $E>0$ such that

$$\|R(t,\tau)f\|\leqslant E|||f||| \tag{2.3}$$

uniformly for $t\in[a,b]$, $\tau\in(0,\delta)$, and

$$\|R(t,\tau)f\|\to0 \qquad \text{as } \tau\to0 \tag{2.4}$$

uniformly on compact subsets of the Banach space $(\mathscr{D},|||\ |||)$ and uniformly in $t\in[a,b]$.

Proof. Let $f\in\mathscr{D}_0$. Then

$$\|R(t,\tau)f\|\leqslant\tau^{-1}\|F(t,\tau)f-f-\tau[Z+L(t)]f\| \\ +\tau^{-1}\|U(t+\tau,t)f-f-\tau[Z+L(t)]f\|. \tag{2.5}$$

The first term on the RHS is bounded by $(D+\max\{C_1,C_2\})|||f|||$ because of conditions (iii) and (C). Now using the identity

$$\{U(t+\tau,t)f-f-\tau[Z+L(t)]f\}=\int_t^{t+\tau}\mathrm{d}r\,[W(r,t)-1][Z+L(t)]f \tag{2.6}$$

the boundness of $W(r,t)$ and the fact that $\mathscr{D}_0$ is dense in $\mathscr{D}$, we obtain (2.3) for all $f\in\mathscr{D}$. Moreover by (ii), (2.6) and remark 3, $R(t,\tau)f$ is strongly convergent to 0 for any $f\in\mathscr{D}_0$ and uniformly in $t\in[a,b]$. Hence by (2.3) we conclude that (2.4) holds for all $f\in\mathscr{D}$ uniformly in t and uniformly on the compact subsets of the Banach space $\mathscr{D}$.

Now we are able to prove the product formula.

Theorem 2.1. Assume that the operators $\{Z+L(t),\ t\in[a,b]\}$ satisfy conditions (A), (B) and (C) and there exists a family of operators $\{F(t,\tau),\ t\in[a,b],\ \tau\in(0,\delta)\}$ for which conditions (i)-(iii) are fulfilled.

Then for any $f\in X$

$$U(t,s)f=\lim_{\substack{N\to\infty\\ \Delta t\to 0}} F(t_1,\Delta t_1)\ldots F(t_N,\Delta t_N)f \tag{2.7}$$

where $t=t_0>t_1\ldots>t_N=s$, $\Delta t_k=t_{k-1}-t_k$, $\Delta t=\max\{\Delta t_k\}$ and $N\Delta t$ is bounded.

Proof. This proof is similar to the proof of the Trotter product formula as given in Reed and Simon (1972).

Let $f\in\mathscr{D}$. Then

$$\|F(t_1,\Delta t_1)\ldots F(t_N,\Delta t_N)f-U(t,s)f\|$$

$$\leqslant\sum_{k=1}^{N}\|F(t_1,\Delta t_1)\ldots F(t_{k-1},\Delta t_{k-1})\{F(t_k,\Delta t_k)-U(t_{k-1},t_k)\}.$$

$$U(t_k,s)f\|\leqslant(N\Delta t)\,\mathrm{e}^{\alpha(t-s)}\sup_{\substack{0<\tau\leqslant\Delta t\\ r,r'\in[s,t]}}\{\tau^{-1}\|[F(r,\tau)-U(r+\tau,r)]U(r',s)f\|\}. \tag{2.8}$$

From remark 3 it follows that the set $\{U(r',s)f,\ r'\in[s,t]\}$ is a compact subset of the Banach space $\mathscr{D}$. Hence, by lemma 2.1, the RHS of (2.7) tends to 0 as $\Delta t\to 0$ for all $f\in\mathscr{D}$. Since $\mathscr{D}$ is dense in X the same holds for all $f\in X$.

3. The convergence theorem in $L^2(\mathbb{R}^\nu)$

Consider the following differential equation in the Hilbert space $L^2(\mathbb{R}^\nu)$:

$$\partial f(x;t)/\partial t=[\Delta+[\tilde{\mathscr{L}}(x,D;t)]f(x;t)] \tag{3.1}$$

where $f(\ ;t)\in L^2(\mathbb{R}^\nu)$, $t\in[a,b]$, Δ denotes the Laplace operator and

$$\tilde{\mathscr{L}}(x,D;t)=\tilde{a}^{\alpha\beta}(x;t)\partial_\alpha\partial_\beta+b^\alpha(x;t)\partial_\alpha+c(x;t). \tag{3.2}$$

We make the following assumptions.

(I) $\tilde{\mathscr{L}}(x,D;t)$ is symmetric on $C_0^\infty(\mathbb{R}^\nu)$ for all $t\in[a,b]$ and

$$a^2=\sup_{\substack{t\in[a,b]\\ x\in\mathbb{R}}}\sum_{\alpha,\beta}(\tilde{a}^{\alpha\beta}(x;t))^2<1.$$

(II) The functions $\tilde{a}^{\alpha\beta}(x;t)$, $b^\alpha(x;t)$ are differentiable with respect to x up to the third order and $c(x;t)$ is continuous. This condition follows from our previous considerations (AM).

(III) The following functions exist and are bounded on $\mathbb{R}^\nu\times[a,b]$ by a common bound $Q<\infty$:

$$\tilde{a}^{\alpha\beta}(\ )\qquad b^\alpha(\ )\qquad c(\ )\qquad \partial_\gamma\tilde{a}^{\alpha\beta}(\ )\partial_\gamma b^\alpha(\ )\qquad \partial_\gamma\partial_\delta a^{\alpha\beta}(\ )$$

$$\partial_t^m\tilde{a}^{\alpha\beta}(\ )\qquad \partial_t^m b^\alpha(\ )\qquad \partial_t^m c(\ )\qquad \text{for } m=1,2.$$

Theorem 3.1. Suppose that $\mathscr{L}(x, D; t) = \Delta + \tilde{\mathscr{L}}(x, D; t)$ satisfies conditions I-III. Then

(a) by remark 3, equation (3.1) defines a family of propagators $U(t, s)$, $t \geq s \in [a, b]$, with properties (*a*)-(*e*) (here $Z \equiv \Delta$, $L(t) \equiv \tilde{\mathscr{L}}(x, D; t)$);

(b) for any $f(\ ; s) \in L^2(\mathbb{R}^\nu)$, $f(\ ; t) = U(t, s)f(\ ; s)$ is given by the path integral expressions (1.3) and (1.4), where the limit is taken in the norm on $L^2(\mathbb{R}^\nu)$ and for any sequence of discretisations $a \leq s = t_N < t_{N-1} < \ldots < t_1 < t_0 = t \leq b$ such that $\Delta t N$ is bounded.

Proof. It is easy to verify that the conditions (A), (B) and (C) are satisfied for $Z \equiv \Delta$, $L(t) \equiv \tilde{\mathscr{L}}(x, D; t)$ and $\mathscr{D}_0 = C_0^\infty(\mathbb{R}^\nu)$ using the assumptions I-III and the arguments of relative boundness as in AM (example (1)). Therefore, by remark 3 statement (a) is true. To prove (b) we have to check that the conditions I-III imply conditions (i)-(iii) with $F(t, \tau)$ given by the following integral:

$$F_{t,\tau}(x|y) = (4\pi\tau)^{-\nu/2}|a(x; t)|^{-1/2} \exp\{\tau c(x; t) - (1/4\tau)a_{\alpha\beta}(x; t)[x^\alpha - y^\alpha + \tau b^\alpha(x; t)][x^\beta - y^\beta + \tau b^\beta(x; t)]\}. \tag{3.3}$$

Indeed the path integral expression (1.4) is an explicit representation of the product formula (2.7) in this case. The conditions (i) and (ii) may be checked exactly in the same way as in AM since by I-III all estimations are uniformly bounded with respect to $t \in [a, b]$. Hence we need to verify condition (iii) only.

Let

$$(K_{t,\tau}f)(x) = \tau^{-1}\left[\left(\int \mathrm{d}y f(y) F_{t,\tau}(x|y)\right) - f(x)\right] \qquad \text{for } f(x) \in C_0^\infty(\mathbb{R}^\nu). \tag{3.4}$$

For a fixed $x \in \mathbb{R}^\nu$ and $t \in [a, b]$ we can transform

$$y^\alpha \to z_\alpha = a_{\alpha\beta}^{1/2}(x, t)(y^\beta - x^\beta - \tau b^\beta(x, t)).$$

Hence (x is fixed)

$$y^\alpha(z; \tau) = x^\alpha + \tau b^\alpha(x; t) + a_{1/2}^{\alpha\beta}(x, t) z_\beta$$

and $(K_{t,\tau}f)(x)$ takes the form

$$(K_{t,\tau}f)(x) = \frac{1}{\tau}\left(\mathrm{e}^{\tau c(x;t)} \int \mathrm{d}z f(y(z; \tau))(4\pi\tau)^{-k/2} \exp(-|z|^2/4\tau) - f(x)\right). \tag{3.5}$$

By the mean-value theorem of differential calculus, for fixed z there exists $\theta \in [0, \tau]$ such that

$$f(y(z; \tau)) - f(y(z; 0)) = \tau b^\alpha(x, t)\partial f(y(z, \theta))/\partial y^\alpha. \tag{3.6}$$

Using now the following identity (valid for smooth enough $\phi(\ )$)

$$\int_{\mathbb{R}^\nu} \mathrm{d}z\, \phi(z) \frac{\exp(-|z|^2/4\tau)}{(4\pi\tau)^{\nu/2}} = \phi(0) + \int_0^\tau \mathrm{d}s \int_{\mathbb{R}^\nu} \mathrm{d}z \frac{\exp(-|z|^2/4s)}{(4\pi s)^{\nu/2}} \Delta\phi(z)$$

with $\phi(z) = f(y(z; 0))$ we obtain

$$(K_{t,\tau}f)(x) = b^\alpha(x; t) \int_{\mathbb{R}^\nu} \mathrm{d}y\, R_{t,\tau}^\theta(x|y)\partial_\alpha f(y) + (1/\tau)(\mathrm{e}^{\tau c(x;t)} - 1)f(x) + \frac{1}{\tau}\int_0^\tau \mathrm{d}s\left(a^{\alpha\beta}(x; t)\int_{\mathbb{R}^\nu} \mathrm{d}y\, R_{s,\tau}(x|y)\partial_\alpha\partial_\beta f(y)\right)$$

where

$$R^{\theta}_{t,\tau}(x|y) = \frac{e^{\tau c(x;t)}}{(4\pi\tau)^{\nu/2}|a(x;t)|^{1/2}} \exp[-(1/4\tau)a_{\alpha\beta}(x;t)(x^{\alpha} - y^{\alpha} + \theta b^{\alpha}(x,t)) \times (x^{\beta} - t^{\beta} + \theta b^{\beta}(x;t))]$$

$\theta \equiv \theta(x, y) \in [0, \tau]$ and $R_{t,\tau}(x|y)$ is obtained from $R^{\theta}_{t,\tau}(x|y)$ by putting $\theta \equiv 0$.

There exist constants M, K and A independent of $x, y \in \mathbb{R}^{\nu}$ and $t \in [a, b]$, $\tau \in (0, \delta)$ such that

$$0 < R^{\theta}_{t,\tau}(x|y) \qquad R_{t,\tau}(x|y) \leq M \frac{1}{(4\pi\tau)^{\nu/2}} \exp[-(A/4\tau)|x-y|^2] \exp(K|x-y|).$$

Hence the norms of operators $R^{\theta}(t;\tau)R(t;\tau)$ are uniformly bounded for $t \in [a, b]$, $\tau \in (0, \delta)$.

Finally, using the relative boundness of the operators $\mu^{\alpha}\partial_{\alpha}$, $X^{\alpha\beta}\partial_{\alpha}\partial_{\beta}$ with respect to $\Delta(\mu^{\alpha}, X^{\alpha\beta} \in \mathbb{R})$ on $C_0^{\infty}(\mathbb{R}^{\nu})$, we obtain the estimation (iii).

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